CONTINUITY AND DIFFERENTIABILITY

Syllabus

1. *Continuity of a function at a point*
2. *Continuity from a left & right*
3. *Continuity of a function in an interval*
4. *Continuous functions*
5. *Discontinuous functions*
6. *Properties of continuous function*
7. *Differentiability of a function and properties*

Total No. of questions in **Continuity and Differentiability** are:

Solved examples.......................................17

Level # 1 91

Level # 2 27

Level # 3 30

Level # 4 31

Total No. of questions…..………………….. **196**

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1. Students are advised to solve the questions of exercises (Levels # 1, 2, 3, 4) in the same sequence or as directed by the faculty members.
2. Level # 3 is not for foundation course students, it will be discussed in fresher and target courses.

Index : Preparing your own list of Important/Difficult Questions

Instruction to fill

(A) Write down the Question Number you are unable to solve in **column A** below, by Pen.

(B) After discussing the Questions written in **column A** with faculties, strike off them in the manner so that you can see at the time of Revision also, to solve these questions again.

(C) Write down the Question Number you feel are important or good in the **column B**.

EXERCISE NO.	COLUMN :A	COLUMN :B
	Questions I am unable to solve in first attempt	Good/Important questions
Level # 1		
Level # 2		
Level # 3		
Level # 4		

Advantages

1. It is advised to the students that they should prepare a question bank for the revision as it is very difficult to solve all the questions at the time of revision.

2. Using above index you can prepare and maintain the questions for your revision.

KEY CONCEPTS

1. Introduction

The word **'Continuous'** means without any break or gap. If the graph of a function has no break or gap or jump, then it is said to be **continuous**.

A function which is not continuous is called a **discontinuous function**.

In other words,

If there is slight (finite) change in the value of a function by slightly changing the value of x then function is continuous, otherwise discontinuous, while studying graphs of functions, we see that graphs of functions sin x, x, cos x, e^x etc. are continuous but greatest integer function [x] has break at every integral point, so it is not continuous. Similarly tan x, cot x, secx, 1/x etc. are also discontinuous function.

For examining continuity of a function at a point, we find its limit and value at that point, If these two exist and are equal, then function is continuous at that point.

2. Continuity of a Function at a Point

A function f(x) is said to be continuous at a point x = a if

(i) f (a) exists

(ii) $\lim_{x \to a} f(x)$ exists and finite

so $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

(iii) $\lim_{x \to a} f(x) = f(a)$.

or function f(x) is continuous at x = a.

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$.

i.e. If right hand limit at 'a' = left hand limit at 'a'= value of the function at 'a'.

If $\lim_{x \to a} f(x)$ does not exist or $\lim_{x \to a} f(x) \neq f(a)$, then f(x) is said to be discontinuous at x= a.

3. Continuity from Left and Right

Function f(x) is said to be

(i) Left continuous at x= a if

$\lim_{x \to a-0} f(x) = f(a)$

(ii) right continuous at x = a if

$\lim_{x \to a+0} f(x) = f(a)$

Thus a function f(x) is continuous at a point x = a if it is left continuous as well as right continuous at x = a.

4. Continuity of a Function in an Interval

(a) A function f(x) is said to be continuous in an open interval (a,b) if it is continuous at every point in (a, b).

For example function y = sin x, y = cos x, y = e^x are continuous in $(-\infty, \infty)$.

(b) A function f(x) is said to be continuous in the closed interval [a, b] if it is-

(i) Continuous at every point of the open interval (a, b).

(ii) Right continuous at x = a.

(iii) Left continuous at x = b.

5. Continuous Functions

A function is said to be continuous function if it is continuous at every point in its domain. Following are examples of some continuous function.

(i) f (x) = x (Identity function)

(ii) f(x) = C (Constant function)

(iii) $f(x) = x^2$

(iv) $f(x) = a_0x^n + a_1x^{n-1} + ... + a^n$ (Polynomial).

(v) f(x) = |x|, x+ |x|, x – |x|, x|x|

(vi) f(x) = sin x, f(x) = cos x

(vii) $f(x) = e^x$, $f(x) = a^x$, a> 0

(viii) f(x) = log x, $f(x) = \log_a x$, a> 0

(ix) f(x) = sinh x, cosh x, tanh x

(x) $f(x) = x^m \sin(1/x)$, m> 0

$f(x) = x^m \cos(1/x)$, m> 0

6. Discontinuous Functions

A function is said to be a discontinuous function if it is discontinuous at at least one point in its domain. Following are examples of some discontinuous function-

(i) $f(x) = 1/x$ at $x = 0$

(ii) $f(x) = e^{1/x}$ at $x = 0$

(iii) $f(x) = \sin 1/x$, $f(x) = \cos 1/x$ at $x = 0$

(iv) $f(x) = [x]$ at every integer

(v) $f(x) = x - [x]$ at every integer

(vi) $f(x) = \tan x$, $f(x) = \sec x$ when $x = (2n+1)\pi/2$, $n \in Z$.

(vii) $f(x) = \cot x$, $f(x) = \text{cosec } x$ when $x = n\pi$, $n \in Z$.

(viii) $f(x) = \coth x$, $f(x) = \text{cosech } x$ at $x = 0$.

7. Properties of Continuous Functions

The sum, difference, product, quotient (If Dr $\neq 0$) and composite of two continuous functions are always continuous functions. Thus if $f(x)$ and $g(x)$ are continuous functions then following are also continuous functions:

(a) $f(x) + g(x)$

(b) $f(x) - g(x)$

(c) $f(x) \cdot g(x)$

(d) $\lambda f(x)$, where λ is a constant

(e) $f(x)/g(x)$, if $g(x) \neq 0$

(f) $f[g(x)]$

Note :

The product of one continuous and one discontinuous function may or may not be continuous.

DIFFERENTIABILITY

8. Differentiability of a Function

A function $f(x)$ is said to be differentiable at a point of its domain if it has a finite derivative at that point. Thus $f(x)$ is differentiable at $x = a$

$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists finitely

$$= \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f'(a-0) = f'(a+0)$

left- hand derivative = Right-hand derivative.

Generally derivative of $f(x)$ at $x = a$ is denoted by $f'(a)$. So $f'(a) = \frac{f(x) - f(a)}{x - a}$

Note : (i) Every differentiable function is necessarily continuous but every continuous function is not necessarily differentiable i.e. Differentiability $\Rightarrow$ continuity

but continuity $\not\Rightarrow$ differentiability

8.1 Differentiability in an interval

(a) A function $f(x)$ is said to be differentiable in an open interval (a,b), if it is differentiable at every point of the interval.

(b) A function $f(x)$ is differentiable in a closed interval $[a,b]$ if it is –

(i) Differentiable at every point of interval (a,b)

(ii) Right derivative exists at $x = a$

(iii) Left derivative exists at $x = b$.

8.2 Differentiable function & their properties

A function is said to be a differentiable function if it is differentiable at every point of its domain.

(a) Example of some differentiable functions:–

(i) Every polynomial function

(ii) Exponential function : a^x, e^x, e^{-x}......

(iii) logarithmic functions : $\log_a x$, $\log_e x$,......

(iv) Trigonometrical functions : $\sin x$, $\cos x$,

(v) Hyperbolic functions : sinhx, coshx,......

(b) Examples of some non– differentiable functions:

(i) $|x|$ at $x = 0$

(ii) $x \pm |x|$ at $x = 0$

(iii) $[x]$, $x \pm [x]$ at every $n \in Z$

(iv) $x \sin\left(\frac{1}{x}\right)$, at $x = 0$

(v) $\cos\left(\frac{1}{x}\right)$, at $x = 0$

(c) The sum, difference, product, quoteint (Dr $\neq 0$) and composite of two differentiable functions is always a differentiable function.