POINT

Preface

The concept of Point is very important for the study of coordinate geometry. This chapter deals with various forms of representing a Point and several associated properties. The concept of coordinates and basics of trigonometry are required to study this chapter.

This book consists of theoretical and practical explanations of all the concepts involved in the chapter. Each article is followed by a ladder of illustration. At the end of the theory part, there are miscellaneous solved examples which involve the application of multiple concepts of this chapter.

Students are advised to go through all these solved examples in order to develop better understanding of the chapter and to have better grasping level in the class.

Total No. of questions in **Point** are :

In Chapter Examples 28

Solved Examples 12

Total no. of questions 40

1. SYSTEM OF CO-ORDINATES

1.1 Cartesian Co-ordinates :

Let XOX′ and YOY′ be two perpendicular straight lines drawn through any point O in the plane of the paper. Then

1.1.1 Axis of x : The line XOX′ is called axis of x.

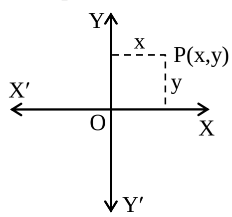
1.1.2 Axis of y : The line YOY′ is called axis of y.

1.1.3 Co-ordinate axes : x axis and y axis together are called axis of co-ordinates or axis of reference.

1.1.4 Origin : The point 'O' is called the origin of co-ordinates or the Origin.

1.1.5 Oblique axis : If both the axes are not perpendicular then they are called as Oblique axes.

1.1.6 Cartesian Co-ordinates : The ordered pair of perpendicular distance from both axis of a point P lying in the plane is called **Cartesian Co-ordinates** of P. If the Cartesian co-ordinates of a point P are (x, y) then x is called abscissa or x coordinate of P and y is called the ordinate or y co-ordinate of point P.



Note :

(i) Co-ordinates of the origin is (0, 0).

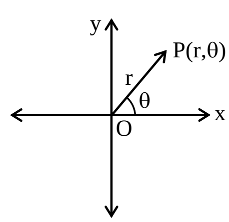
(ii) y co-ordinate on x- axis is zero.

(iii) x co-ordinate on y- axis is zero.

1.2 Polar Co-ordinates :

Let OX be any fixed line which is usually called the initial line and O be a fixed point on it. If distance of any point P from the pole O is 'r' and ∠ XOP = θ, then (r, θ) are called the polar co-ordinates of a point P.

If (x, y) are the Cartesian co-ordinates of a point P, then



$$x = r\cos\theta \ ; \qquad y = r\sin\theta$$

and $$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example based on System of Co-ordinates

Ex.1 If Cartesian co-ordinates of any point are $(\sqrt{3}, 1)$ then its polar co-ordinates is -

(A) $(2, \pi/3)$ (B) $(\sqrt{2}, \pi/6)$

(C) $(2, \pi/6)$ (D) None of these

Sol. $\sqrt{3} = r\cos\theta$, $1 = r\sin\theta$

$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2.$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6 \Rightarrow (2, \pi/6)$

Ans. [C]

Ex.2 If polar coordinates of any points are $(2, \pi/3)$ then its Cartesian coordinates is -

(A) $(1, -\sqrt{3})$ (B) $(1, \sqrt{3})$

(C) $(\sqrt{3}, 1)$ (D) None of these

Sol. $x = 2\cos\pi/3$, $y = 2\sin\pi/3$

$= 1; = \sqrt{3} \Rightarrow (1, \sqrt{3})$ **Ans. [B]**

2. DISTANCE FORMULA

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Note :

(i) Distance of a point P(x,y) from the origin $= \sqrt{x^2 + y^2}$

(ii) Distance between two polar co-ordinates $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ is given by

$$AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

Example based on Distance Formula

Ex.3 Find the distance between P(3,–2) and Q(–7,–5).

Sol. $PQ = \sqrt{(3+7)^2 + (-2+5)^2}$

$= \sqrt{100+9} = \sqrt{109}$ **Ans.**

Ex.4 Distance of a point P(8, 6) from origin is

$= \sqrt{8^2+6^2} = \sqrt{100} = 10$

Ex.5 Find the distance between

$P(2, -\frac{\pi}{6})$ and $Q(3, \frac{\pi}{6})$

Sol. $PQ = \sqrt{2^2+3^2-2.2.3.\cos(-\frac{\pi}{6}-\frac{\pi}{6})}$

$= \sqrt{13-12\times\cos(-\frac{\pi}{3})} = \sqrt{13-12\times\frac{1}{2}} = \sqrt{7}$ **Ans.**

Ex.6 Distance between points (a, 0) and (0, a) is

(A) $\sqrt{2}$ a (B) $2a^2$

(C) 2a (D) $2\sqrt{2}$ a

Sol. $D = \sqrt{(a-0)^2+(0-a)^2}$

$= \sqrt{a^2+a^2} = \sqrt{2a^2} = \sqrt{2}$ a **Ans. [A]**

Ex.7 If distance between the point (x, 2) and (3, 4) is 2, then the value of x is -

(A) 1 (B) 2 (C) 3 (D) 0

Sol. $2 = \sqrt{(x-3)^2+(2-4)^2} \Rightarrow 2 = \sqrt{(x-3)^2+4}$

Squaring both sides

$4 = (x-3)^2 + 4 \Rightarrow x - 3 = 0 \Rightarrow x = 3$

Ans. [C]

Ex.8 The point whose abscissa is equal to its ordinate and which is equidistant from the point (1,0) and B(0,3) is -

(A) (3, 3) (B) (2, 2)

(C) (1,1) (D) (4, 4)

Sol. Let the point P(k, k)

given $\Rightarrow$ PA = PB

$\Rightarrow \sqrt{(k-1)^2+k^2} = \sqrt{k^2+(k-3)^2}$

$\Rightarrow 2k^2 - 2k + 1 = 2k^2 - 6k + 9$

$\Rightarrow 4k = 8 \Rightarrow k = 2$ **Ans. [B]**

3. APPLICATIONS OF DISTANCE FORMULA

3.1 Position of Three Points :

Three given points A, B, C are collinear, when sum of any two distance out of AB, BC, CA is equal to remaining third distance. Otherwise the points will be vertices of a triangles.

3.1.1 Types of Triangle : If A, B and C are vertices of triangle then it would be.

(a) Equilateral triangle, when AB = BC = CA.

(b) Isosceles triangle, when any two distance are equal.

(c) Right angle triangle, when sum of square of any two distances is equal to square of the third distance.

3.2 Position of four Points :

Four given point A, B, C and D are vertices of a

(a) Square if AB = BC = CD = DA and AC = BD

(b) Rhombus if AB = BC = CD = DA & AC $\neq$ BD

(c) Parallelogram if AB = DC; BC = AD; AC $\neq$ BD

(d) Rectangle if AB = CD; BC = DA; AC = BD

	Quadrilateral	Diagonals	Angle between diagonals
(i)	Parallelogram	Not equal	$\theta \neq \frac{\pi}{2}$
(ii)	Rectangle	Equal	$\theta \neq \frac{\pi}{2}$
(iii)	Rhombus	Not equal	$\theta = \frac{\pi}{2}$
(iv)	Square	Equal	$\theta = \frac{\pi}{2}$

Note :

(i) Diagonal of square, rhombus, rectangle and parallelogram always bisect each other.

(ii) Diagonal of rhombus and square bisect each other at right angle.

(iii) Four given points are collinear, if area of quadrilateral is zero.

Example based on **Applications of Distance Formula**

Ex.9 The point A(8,2); B(5, –3) and C(0, 0) are vertices of

(A) An equilateral triangle

(B) A right angled triangle

(C) An isosceles right angled triangle

(D) An isosceles triangle

Sol. $AB = \sqrt{(8-5)^2+(2+3)^2} = \sqrt{34}$

$BC = \sqrt{5^2+(-3)^2} = \sqrt{34}$

$CA = \sqrt{8^2+2^2} = \sqrt{68}$

as $AB^2 + BC^2 = 34 + 34 = CA^2$ and

AB = BC Hence the given vertices are of an isosceles right angled triangle. **Ans. [C]**

Ex.10 Show that the points A(1,1), B(–2, 7) and C(3, –3) are collinear

Sol. AB = $\sqrt{(1+2)^2+(1-7)^2} = \sqrt{9+36} = 3\sqrt{5}$

BC = $\sqrt{(-2+5)^2+(7+3)^2} = \sqrt{25+100} = 5\sqrt{5}$

CA = $\sqrt{(3-1)^2+(-3-1)^2} = \sqrt{4+16} = 2\sqrt{5}$

Clearly BC = AB + AC. Hence A, B, C are collinear

4. SECTION FORMULA

Co-ordinates of a point which divides the line segment joining two points P(x_1, y_1) and Q(x_2, y_3) in the ratio $m_1 : m_2$ are.

(i) For internal division

$$=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

(ii) For external division

$$=\left(\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}\right)$$

(iii) Co-ordinates of mid point of PQ are

put $m_1 = m_2$; $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Note :

(i) Co-ordinates of any point on the line segment joining two points P(x, y) and Q(x_2, y_2) are

$$\left(\frac{x_1+\lambda x_2}{2}, \frac{y_1+\lambda y_2}{2}\right), (\lambda \neq -1)$$

(ii) Lines joins (x_1, y_1) and (x_2, y_2) is divided by

(a) x axis in the ratio = $-y_1 / y_2$

(b) y axis in the ratio = $-x_1 / x_2$

if ratio is positive divides internally, if ratio is negative divides externally.

(iii) Line ax + by + c = 0 divides the line joining the points (x_1, y_2) and (x_1, y_2) in the ratio

$$-\left(\frac{ax_1+by_1+c}{ax_2+by_2+c}\right)$$

Example based on **Section Formula**

Ex.11 Find the co–ordinates of point of internal and external division of the line segment joining two points (3, –1) and (3, 4) in the ratio 2 :3

Sol. Internal division

$$x = \frac{2(3)+3(3)}{2+3} = 3$$

$$y = \frac{2(4)+3(-1)}{2+3} = 1$$

Hence point (3, 1) **Ans.**

External division $x = \frac{2(3)-3(3)}{2-3} = 3$

Hence point (3, –11) **Ans.**

Ex.12 Mid points of (2, 3) and (6, 7) is

$\left(\frac{2+6}{2}, \frac{3+7}{2}\right)$ = (4, 5) **Ans.**

Ex.13 Ratio in which the line 3x + 4y = 7 divides the line segment joining the points (1,2) and (–2, 1) is -

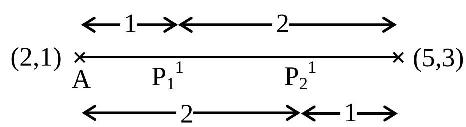
Sol. $= \frac{3(1)+4(2)-7}{3(-2)+4(1)-7} = -\frac{4}{-9} = \frac{4}{9}$ **Ans.**

Ex.14 The points of trisection of line joining the points A (2, 1) and B (5, 3) are

(A) $\left(4, \frac{5}{3}\right)\left(3, \frac{7}{3}\right)$ (B) $\left(3, \frac{7}{3}\right)\left(\frac{5}{3}, 4\right)$

(C) $\left(3, \frac{5}{3}\right)\left(4, \frac{7}{3}\right)$ (D) $\left(4, \frac{7}{3}\right)\left(3, \frac{7}{3}\right)$

Sol.



$$P_1(x,y) = \left(\frac{1\times5+2\times2}{1+2}, \frac{1\times3+2\times1}{1+2}\right) = \left(3, \frac{5}{3}\right)$$

$$P_2(x,y) = \left(\frac{1\times5+1\times2}{1+2}, \frac{2\times3+1\times1}{1+2}\right) = \left(4, \frac{7}{3}\right)$$

Ans. [C]

Ex.15 The ratio in which the lines joining the (3, –4) and (–5, 6) is divided by x-axis

(A) 2 : 3 (B) 6 : 4

(C) 3 : 2 (D) None

Sol. $= -\left(\frac{-4}{6}\right)$ = 2 : 3 **Ans. [A]**

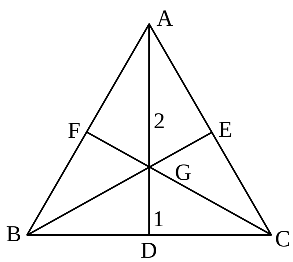
5. CO-ORDINATE OF SOME PARTICULAR

POINTS

Let $A(x_1,y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

5.1 Centroid :

The centroid is the point of intersection of the medians (Line joining the mid point of sides and opposite vertices).

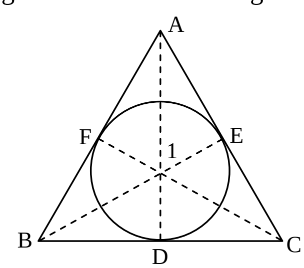


Centroid divides the median in the ratio of 2 : 1. Co-ordinates of centroid

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

5.2 Incentre :

The incentre of the point of intersection of internal bisector of the angle. Also it is a centre of a circle touching all the sides of a triangle.



Co-ordinates of incentre

$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$ where a, b, c are the sides of triangle ABC.

Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides eg.

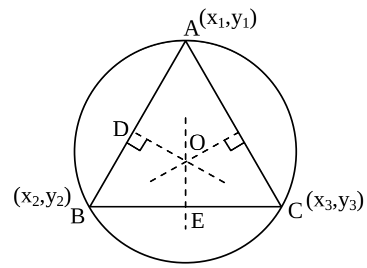
$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$$

(ii) Incentre divides the angle bisectors in the ratio (b + c):a, (c + a):b, and (a + b):c

(iii) **Excentre :** Point of intersection of one internal angle bisector and other two external angle bisector is called as excentre. There are three excentre in a triangle. Co-ordinate of each can be obtained by changing the sign of a, b, c respectively in the formula of In centre.

5.3 Circumcentre :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. It is also the centre of a circle passing vertices of the triangle. If O is the circumcentre of any triangle ABC, then $OA^2 = OB^2 = OC^2$

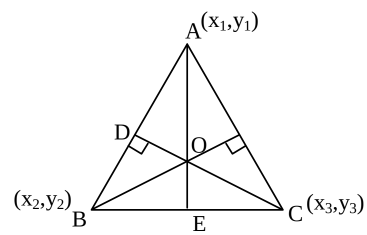


Note :

If a triangle is right angle, then its circumcentre is the mid point of hypotenuse.

5.4 Ortho Centre :

It is the point of intersection of perpendicular drawn from vertices on opposite sides (called altitudes) of a triangle and can be obtained by solving the equation of any two altitudes.



Note :

If a triangle is right angle triangle, then orthocentre is the point where right angle is formed.

Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocentre, circumcentre, coincides

(ii) Ortho centre, centroid and circumcentre are always colinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

Example based on **Coordinates of Some Particular Points**

Ex.16 Centroid of the triangle whose vertices are (0, 0), (2, 5) and (7, 4) is

$\left(\frac{0+2+7}{3}, \frac{0+5+4}{3}\right)$ = (3, 3) **Ans.**

Ex.17 Incentre of triangle whose vertices are A (–36, 7) B (20, 7) C (0, –8) is -

Sol. Using distance formula

$a = BC = \sqrt{20^2 + (7+8)^2} = 25$

$b = CA = \sqrt{36^2 + (7+8)^2} = 39$

$c = AB = \sqrt{(36+20)^2 + (7-7)^2} = 56$

$$I = \left(\frac{25(-36)+39(20)+56(0)}{25+39+56}, \frac{25(7)+39(7)+56(-8)}{25+39+56}\right)$$

$I = (-1, 0)$ **Ans.**

Ex.18 If (1,4) is the centroid of a triangle and its two vertices are (4,–3) and (–9,7) then third vertices is -

(A) (7, 8) (B) (8, 7)

(C) (8, 8) (D) (6, 8)

Sol. Let the third vertices of triangle be (x, y) then

$1 = \frac{x+4-9}{3} \Rightarrow x = 8$

$4 = \frac{y-3+7}{3} \Rightarrow y = 8$ **Ans. [C]**

Ex.19 If (0, 1), (1, 1) and (1, 0) are middle points of the sides of a triangle, then its incentre is -

(A) $(2-\sqrt{2}, -2+\sqrt{2})$

(B) $(2-\sqrt{2}, 2-\sqrt{2})$

(C) $(2+\sqrt{2}, 2+\sqrt{2})$

(D) $(2+\sqrt{2}, -2-\sqrt{2})$

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then

$x_1 + x_2 = 0$, $x_2 + x_3 = 2$, $x_3 + x_1 = 2$

$y_1 + y_2 = 2$, $y_2 + y_3 = 2$, $y_3 + y_1 = 0$

Solving these equations, we get

A(0, 0), B(0, 2) and C(2, 0)

Now

$a = BC = 2\sqrt{2}$, $b = CA = 2$, $c = AB = 2$

Thus incentre of Δ ABC is

$(2-\sqrt{2}, 2-\sqrt{2})$ **Ans. [B]**

6. AREA OF TRIANGLE AND QUADRILATERAL

6.1 Area of Triangle

Let $A(x_1,y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then -

$$\text{Area of Triangle ABC} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Note :

(i) If area of a triangle is zero, then the points are collinear.

(ii) In an equilateral triangle

(a) having sides 'a' area is $= \frac{\sqrt{3}}{4}a^2$

(b) having length of perpendicular as 'p' area is $\frac{p^2}{\sqrt{3}}$

(iii) If a triangle has polar co-ordinate (r,θ_1), (r_2, θ_2) and (r_3, θ_3) then its area

$$\Delta = \frac{1}{2}[r_1r_2 \sin(\theta_2 - \theta_1) + r_2r_3 \sin(\theta_3 - \theta_2) + r_3r_1 \sin(\theta_1 - \theta_3)]$$

6.2 Area of quadrilateral :

If (x_1,y_1), (x_2, y_2), (x_3, y_3) and (x_4, x_4) are vertices of a quadrilateral then its area

$$= \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)]$$

Note :

(i) If the area of quadrilateral joining four points is zero then those four points are colinear.

(ii) If two opposite vertex of rectangle are (x_1, y_1) and (x_2, y_2) and sides are parallel to coordinate axes then its area is

$= |(y_2 - y_1)(x_2 - x_1)|$

(iii) If two opposite vertex of a square are A (x_1, y_1) and C (x_2, y_2) then its area is

$$= \frac{1}{2}AC^2 = \frac{1}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

Example based on **Area of Triangle & Quadrilateral**

Ex.20 If the vertices of a triangle are (1, 2) (4, –6) and (3, 5) then its area is -

$\Delta = \frac{1}{2}[1(-6-5) + 4(5-2) + 3(2+6)]$

$= \frac{1}{2}[-11 + 12 + 24]$

$= \dfrac{25}{2}$ square unit **Ans.**

Ex.21 If (1,1) (3,4) (5, –2) and (4, –7) are vertices of a quadrilateral then its area

$= \dfrac{1}{2}[1 \times 4 - 3 \times 1 + 3 \times (-2) - 5(4) + 5(-7)$

$- 4(-2) + 4(1) - 1(-7)]$

$= \dfrac{1}{2}[4 - 3 - 6 - 20 - 35 + 8 + 4 + 7]$

$= \dfrac{41}{2}$ units. **Ans.**

Ex.22 If the coordinates of two opposite vertex of a square are (a, b) and (b, a) then area of square is -

(A) $(a - b)^2$ (B) $a^2 + b^2$

(C) $2(a - b)^2$ (D) $(a + b)^2$

Sol. We know that Area of square = $\dfrac{1}{2}d^2$

$= \dfrac{1}{2}[(a - b)^2 + (b - a)^2]$

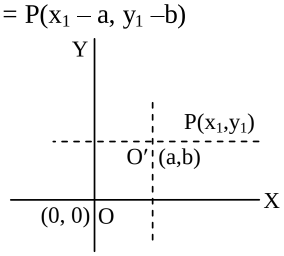
$= (a - b)^2$ **Ans. [A]**

7. TRANSFORMATION OF AXES

7.1 Parallel transformation :

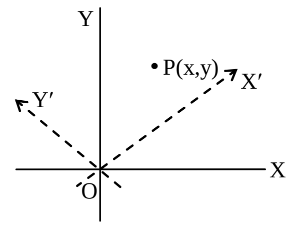
Let origin O(0, 0) be shifted to a point (a, b) by moving the x axis and y axis parallel to themselves. If the co-ordinate of point P with reference to old axis are (x_1, y_1) then co-ordinate of this point with respect to new axis will be $(x_1 - a, y_1 - b)$

$P(x', y') = P(x_1 - a, y_1 - b)$



7.2 Rotational transformation :

Let OX and OY be the old axis and OX′ and OY′ be the new axis obtained by rotating the old OX and OY through an angle θ.



Again, if co-ordinates of any point P(x, y) with reference to new axis will be (x′, y′), then

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

The above relation between (x, y) and (x′, y′) can be easily obtained with the help of following table.

	$x\downarrow$	$y\downarrow$
$x'\rightarrow$	$\cos\theta$	$\sin\theta$
$y'\rightarrow$	$-\sin\theta$	$\cos\theta$

7.3 Reflection (Image) of a Point :

Let (x, y) be any point, then its image w.r.t.

(i) x-axis $\Rightarrow$ (x, –y)

(ii) y-axis $\Rightarrow$ (–x, y)

(iii) origin $\Rightarrow$ (–x , –y)

(iv) line y = x $\Rightarrow$ (y, x)

Example based on Transformation of Axes

Ex.23 If axis are transformed from origin to the point (–2, 1) then new co-ordinates of (4, –5) is -

(A) (6, 4) (B) (2, – 6)

(C) (6, – 6) (D) (2, – 4)

Sol. [4 – (– 2), – 5 – 1] = (6, – 6) **Ans. [C]**

Ex.24 Keeping the origin constant axis are rotated at an angle 30º in negative direction then coordinate of (2,1) with respect to old axis is -

(A) $\left(\dfrac{2+\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}\right)$

(B) $\left(\dfrac{2\sqrt{3}+1}{2}, \dfrac{-2+\sqrt{3}}{2}\right)$

(C) $\left(\dfrac{2\sqrt{3}+1}{2}, \dfrac{2-\sqrt{3}}{2}\right)$

(D) None of these

Sol.

	x	y
2	$\cos(-30^o)$	$\sin(-30^o)$
1	$-\sin(-30^o)$	$\cos(-30^o)$

$x = 2\cos30^o + \sin30^o = \dfrac{2\sqrt{3}+1}{2}$

$y = -2\sin30^o + \cos30^o = \dfrac{-2+\sqrt{3}}{2}$ **Ans. [B]**

Ex.25 If the new coordinates of a point after the rotation of axis in the negative direction by an angle of $\pi/3$ are (4, 2) then coordinate with respect to old axis are -

(A) $(-2\sqrt{3} + 1, 2 + \sqrt{3})$

(B) $(2 + \sqrt{3}, -2\sqrt{3} - 1)$

(C) $(2 + \sqrt{3}, -2\sqrt{3} + 1)$

(D) $(2 - \sqrt{3}, -2\sqrt{3} - 1)$

Sol.

	x	y
4	$\cos(-60^o)$	$\sin(-60^o)$
2	$-\sin(-60^o)$	$\cos(-60^o)$

$\therefore\ x = x'\cos\alpha - y'\sin\alpha$

$= 4\cos(-60) - 2\sin(-60) = 2 + \sqrt{3}$

$y = x'\sin\alpha - y'\cos\alpha$

$= 4\sin(-60) + 2\cos(-60) = -2\sqrt{3} + 1$

Hence the coordinates are

$(2 + \sqrt{3}, -2\sqrt{3} + 1)$. **Ans. [C]**

8. LOCUS

A locus is the curve traced out by a point which moves under certain geometrical conditions. To find a locus of a point first we assume the Co-ordinates of the moving point as (h, k) then try to find a relation between h and k with the help of the given conditions of the problem. In the last we replace h by x and k by y and get the locus of the point which will be an equated between x and y.

Note :

(i) Locus of a point P which is equidistant from the two point A and B is straight line and is a perpendicular bisector of line AB.

(ii) In above case if

PA = kPB where $k \neq 1$

then the locus of P is a circle.

(iii) Locus of P if A and B is fixed.

(a) Circle if $\angle APB$ = Constant

(b) Circle with diameter AB if $\angle ABB = \dfrac{\pi}{2}$

(c) Ellipse if PA + PB = Constant

(d) Hyperbola if PA – PB = Constant

Example based on Locus

Ex.26 The locus of a point which is equidistant from point (6, –1) and (2, 3)

Sol. Let the point is (h, k) then

$\sqrt{(x-6)^2+(k+1)^2} = \sqrt{(h-2)^2+(k-3)^2}$

$\Rightarrow h - k = 3$

Hence locus is x – y = 3 **Ans.**

Ex.27 Find the locus of a point such that the sum of its distance from the points (0, 2) and (0, –2) is 6.

Sol. Let P (h, k) be any point on the locus and let A(0,2) and B(0, –2) be the given points.

By the given condition PA + PB = 6

$\Rightarrow \sqrt{(h-0)^2+(k-2)^2} + \sqrt{(h-0)^2+(k+2)^2} = 6$

$\Rightarrow \sqrt{h^2+(k-2)^2} = 6 - \sqrt{h^2+(k+2)^2}$

$\Rightarrow h^2 + (k+2)^2 = 36 - 12\sqrt{h^2+(k+2)^2} + h^2 + (k+2)^2$

$\Rightarrow -8k - 36 = -12\sqrt{h^2+(k+2)^2}$

$\Rightarrow (2k+9)^2 = 9(h^2 + (k+2)^2)$

$\Rightarrow 4k^2 + 36k + 81 = 9h^2 + 9k^2 + 36k + 36$

$\Rightarrow 9h^2 + 5k^2 = 45$

Hence , locus of (h, k) is $9x^2 + 5y^2 = 45$

Ans.

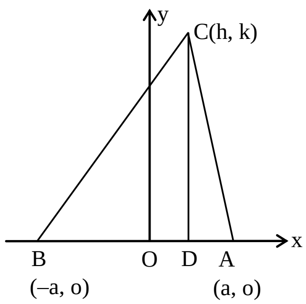
Ex.28 A (a,0) and B (–a, 0) are two fixed points of Δ ABC. If its vertex C moves in such way that cotA + cotB = λ, where λ is a constant, then the locus of the point C is –

(A) $y\lambda = 2a$ (B) $y = \lambda a$

(C) $ya = 2\lambda$ (D) None of these

Sol. We may suppose that coordinates of two fixed points A, B are (a, 0) and (–a, 0) and variable point C is (h, k).

From the adjoining figure



$\cot A = \dfrac{DA}{CD} = \dfrac{a-h}{k}$

$\cot B = \dfrac{BD}{CD} = \dfrac{a+h}{k}$

But cot A + cot B =λ, so we have

$\dfrac{a-h}{k} + \dfrac{a+h}{k} = \lambda \Rightarrow \dfrac{2a}{k} = \lambda$

Hence locus of C is $y\lambda = 2a$ **Ans. [A]**

9. SOME IMPORTANT POINTS

(i) Quadrilateral containing two sides parallel is called as Trapezium whose area is given by $\frac{1}{2}$ (sum of parallel sides) × (Distance between parallel sides)

(ii) A triangle having vertices $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$, then area is
$\Delta = a^2[(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)]$

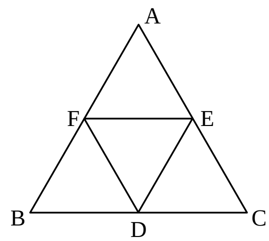
(iii) Area of triangle formed by Co-ordinate axis and the line ax + by + c = 0 is equal to $\dfrac{c^2}{2ab}$

(iv) When x co-ordinate or y co-ordinate of all vertex of triangle are equal then its area is zero.

(v) In a Triangle ABC, of D, E, F are midpoint of sides AB, BC and CA then

$EF = \dfrac{1}{2} BC$ and

$\Delta DEF = \dfrac{1}{4} (\Delta ABC)$



(vi) Area of Rhombus formed by
$ax \pm by \pm c = 0$ is $\dfrac{2c^2}{ab}$

(vii) Three points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear if

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

(viii)When one vertex is origin then area of triangle

$$\frac{1}{2} = (x_1y_2 - x_2y_1)$$

(ix) To remove the term of xy in the equation $ax^2 + 2hxy + by^2 = 0$, the angle θ through which the axis must be turned (rotated) is given by

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$

SOLVED EXAMPLES

Ex.1 The point A divides the join of the points (–5,1) and (3,5) in the ratio k : 1 and coordinates of points B and C are (1, 5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals -

(A) 7, 9 (B) 6, 7

(C) 7, 31/9 (D) 9, 31/9

Sol. $A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of ΔABC = 2 units

$$\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2)+1\left(-2-\frac{5k+1}{k+1}\right) + 7\left(\frac{5k+1}{k+1}-5\right)\right] = \pm 2$$

$\Rightarrow$ 14k – 66 = ± 4 (k+1)

$\Rightarrow$ k = 7 or 31/9 **Ans. [C]**

Ex.2 The vertices of a triangle are A(0, –6), B (–6, 0) and C (1,1) respectively, then coordinates of the ex-centre opposite to vertex A is -

(A) (–3/2, –3/2) (B) (–4, 3/2)

(C) (–3/2, 3/2) (D) (–4, –6)

Sol. $a = BC = \sqrt{(-6-1)^2+(0-1)^2} = \sqrt{50} = 5\sqrt{2}$

$b = CA = \sqrt{(1-0)^2+(1+6)^2} = \sqrt{50} = 5\sqrt{2}$

$c = AB = \sqrt{(0+6)^2+(-6-0)^2} = \sqrt{72} = 6\sqrt{2}$

coordinates of Ex-centre opposite to vertex A are

$$x = \frac{-ax_1+bx_2+cx_3}{-a+b+c}$$

$$= \frac{-5\sqrt{2}.0+5\sqrt{2}(-6)+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2+6\sqrt{2}}}$$

$$= \frac{-24\sqrt{2}}{6\sqrt{2}} = -4$$

$$y = \frac{-ay_1+by_2+cy_3}{-a+b+c}$$

$$= \frac{-5\sqrt{2}(-6)+5\sqrt{2}.0+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2+6\sqrt{2}}} = \frac{-36\sqrt{2}}{6\sqrt{2}} = -6$$

Hence coordinates of ex-centre are (–4, –6)

Ans. [D]

Ex.3 If the middle point of the sides of a triangle ABC are (0, 0); (1, 2) and (–3, 4), then the area of triangle is –

(A) 40 (B) 20

(C) 10 (D) 60

Sol. If the given mid points be D, E, F; then the area of ΔDEF is given by

$\Rightarrow \frac{1}{2}$ [0(2 – 4) + 1(4 – 0) –3(0 – 2)]

$\Rightarrow \frac{1}{2}$ [0 + 4 + 6] = 5

∴ Area of the triangle ABC = 4 × 5 = 20

Ans. [B]

Ex.4 The three vertices of a parallelogram taken in order are (–1, 0), (3, 1) and (2, 2) respectively. Find the coordinate of the fourth vertex -

(A) (2, 1) (B) (–2, 1)

(C) (1, 2) (D) (1, –2)

Sol. Let A(–1, 0) , B(3, 1) , C(2, 2) and D(x, y) be the vertices of a parallelogram ABCD taken in order. Since, the diagonals of a parallelogram bisect each other.

∴ Coordinates of the mid point of AC = Coordinates of the mid-point of BD

$$\Rightarrow \left(\frac{-1+2}{2}, \frac{0+2}{2}\right) = \left(\frac{3+x}{2}, \frac{1+y}{2}\right)$$

$$\Rightarrow \left(\frac{1}{2}, 1\right) = \left(\frac{3+x}{2}, \frac{y+1}{2}\right)$$

$\Rightarrow \frac{3+x}{2} = \frac{1}{2}$ and $\Rightarrow \frac{y+1}{2} = 1$

$\Rightarrow$ x = –2 and y = 1.

Hence the fourth vertex of the parallelogram is (–2, 1) **Ans. [B]**

Ex.5 Which of the following statement is true ?

(A) The Point A(0, –1), B(2,1), C(0,3) and D(–2, 1) are vertices of a rhombus

(B) The points A(–4, –1), B(–2, –4), C(4, 0) and D(2, 3) are vertices of a square

(C) The points A(–2, –1), B(1, 0), C(4, 3) and D(1, 2) are vertices of a parallelogram

(D) None of these

Sol. Here (i) A(0,–1), B(2,1), C(0,3), D(–2,1) for a rhombus all four sides are equal but the diagonal are not equal, we see AC = $\sqrt{0+4^2}$ =4, BD = $\sqrt{4^2-0}$ = 4

Hence it is a square, not rhombus

(ii) Here AB = $\sqrt{2^2+3^2} = \sqrt{13}$,

BC = $\sqrt{6^2+4^2} = \sqrt{52}$

AB ≠ BC Hence not square.

(iii) In this case mid point of AC is

$\left(\frac{4-2}{2}, \frac{3-1}{2}\right)$ or (1,1)

Also midpoint of diagonal BD $\left(\frac{1+1}{2}, \frac{0+2}{2}\right)$ or (1, 1)

Hence the point are vertices of a parallelogram.

Ans. [C]

Note : The students should note that the squares, rhombus and the rectangle are also parallelograms but every parallelogram is not square etc. The desired answer should be pinpointed carefully.

Ex.6 The condition that the three points (a, 0), $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are collinear if -

(A) $t_1 + t_2 = 0$ (B) $t_1t_2 = 2$

(C) $t_1t_2 = -1$ (D) None of these

Sol. Here the points are collinear if the area of the triangle is zero.

Hence

$1/2\ [a(t_1^2 - 1)2at_2 - 2at_1(at_2^2 - a)] = 0$

or $t_2(t_1^2 - 1) - t_1(t_2^2 - 1) = 0$

$\Rightarrow$ $t_2t_1^2 - t_2 - t_1t_2^2 + t_1 = 0$

$\Rightarrow$ $(t_1 - t_2)(t_1t_2 + 1) = 0,\ t_1 \neq t_2$

$\therefore$ $t_1t_2 + 1 = 0 \Rightarrow t_1t_2 = -1$

Ans. [C]

Note : The students should note that the points lie on the parabola $y^2 = 4ax$, and (a,0) is focus, the condition $t_1t_2 = -1$ is well known condition for the extremities of a focal chord, as we shall see in parabola in our further discussions.

Ex.7 If the origin is shifted to (1, –2) and axis are rotated through an angle of 30º the co-ordinate of (1,1) in the new position are –

(A) $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ (B) $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

(C) $\left(\frac{3}{2}, -\frac{3\sqrt{2}}{2}\right)$ (D) None of these

Sol. If coordinates are (x′, y′) then

$x = h + x'\cos\alpha - y'\sin\alpha$.

$y = k + x'\sin\alpha + y'\cos\alpha$

Where,

(x, y) = (1, 1) , (h, k) = (1, –2), α = 30º

$\therefore$ $1 = 1 + x'\cos 30 - y'\sin 30$

$\Rightarrow$ $x'\sqrt{3} - y' = 0$

and $1 = -2 + x'\sin 30 + y'\cos 30$

$\Rightarrow$ $3 = \frac{x'+y'\sqrt{3}}{2}$

$\Rightarrow$ $x' = \frac{3}{2},\ y' = \frac{3\sqrt{3}}{2}$ **Ans. [B]**

Ex.8 The locus of the point, so that the join of (–5, 1) and (3, 2) subtends a right angle at the moving point is

(A) $x^2 + y^2 + 2x - 3y - 13 = 0$

(B) $x^2 - y^2 + 2x + 3y - 13 = 0$

(C) $x^2 + y^2 - 2x + 3y - 13 = 0$

(D) $x^2 + y^2 - 2x - 3y - 13 = 0$

Sol. Let P (h, k) be moving point and let A(–5, 1) and B(3,2) be given points.

By the given condition ∠APB =90º

$\therefore$ Δ APB is a right angled triangle.

$\Rightarrow$ $AB^2 = AP^2 + PB^2$

$\Rightarrow$ $(3+5)^2 + (2-1)^2 = (h+5)^2 + (k-1)^2 + (h-3)^2 + (k-2)^2$

$\Rightarrow$ $65 = 2(h^2 + k^2 + 2h - 3k) + 39$

$\Rightarrow$ $h^2 + k^2 + 2h - 3k - 13 = 0$

Hence locus of (h, k) is

$x^2 + y^2 + 2x - 3y - 13 = 0$ **Ans. [A]**

Ex.9 The ends of the rod of length ℓ moves on two mutually perpendicular lines, find the locus of the point on the rod which divides it in the ratio $m_1 : m_2$

(A) $m_1^2 x^2 + m_2^2 y^2 = \dfrac{\ell^2}{(m_1+m_2)^2}$

(B) $(m_2 x)^2 + (m_1 y)^2 = \left(\dfrac{m_1 m_2 \ell}{m_1+m_2}\right)^2$

(C) $(m_1 x)^2 + (m_2 y)^2 = \left(\dfrac{m_1 m_2 \ell}{m_1+m_2}\right)^2$

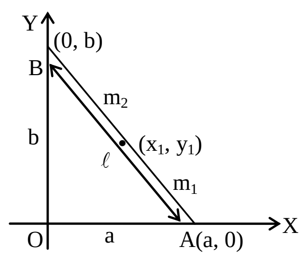
(D) None of these

Sol. Let (x_1, y_1) be the point that divide the rod AB = ℓ, in the ratio $m_1 : m_2$, and OA = a, OB = b say

$\therefore\ a^2 + b^2 = \ell^2$ ...(1)

Now $x_1 = \left(\dfrac{m_2 a}{m_1+m_2}\right) \Rightarrow a = \left(\dfrac{m_1+m_2}{m_2}\right)x_1$

$y_1 = \left(\dfrac{m_2 b}{m_1+m_2}\right) \Rightarrow b = \left(\dfrac{m_1+m_2}{m_1}\right)y_1$



These putting in (1)

$$\frac{(m_1+m_2)^2}{m_2^2}x_1^2 + \frac{(m_1+m_2)^2}{m_1^2}y_1^2 = \ell^2$$

$\therefore$ Locus of (x_1, y_1) is

$m_1^2 x^2 + m_2^2 y^2 = \left(\dfrac{m_1 m_2 \ell}{m_1+m_2}\right)^2$ **Ans. [C]**

Ex.10 A point P moves such that the sum of its distance from (ae, 0) and (–ae, 0)) is always 2a then locus of P is (when $0 < e < 1$)

(A) $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1-e^2)} = 1$

(B) $\dfrac{x^2}{a^2} - \dfrac{y^2}{a^2(1-e^2)} = 1$

(C) $\dfrac{x^2}{a^2(1-e^2)} + \dfrac{y^2}{a^2} - = 1$

(D) None of these

Sol. Let P(h, k) be the moving point such that the sum of its distance from A(ae, 0) and B(–ae, 0) is 2a.

Then , PA + PB = 2a

$\Rightarrow \sqrt{(h-ae)^2+(k-0)^2} + \sqrt{(h+ae)^2+(k-0)^2} = 2a$

$\Rightarrow \sqrt{(h-ae)^2+k^2} = 2a - \sqrt{(h+ae)^2+(k-0)^2}$

Squaring both sides, we get

$\Rightarrow\ (h-ae)^2 + k^2 = 4a^2 + (h+ae)^2 + k^2 - 4a\sqrt{(h+ae)^2+k^2}$

$\Rightarrow\ -4aeh - 4a^2 = -4a\sqrt{(h+ae)^2+k^2}$

$\Rightarrow (eh + a) = \sqrt{(h+ae)^2+k^2}$

[Squaring both sides]

$\Rightarrow (eh+a)^2 = (h+ae)^2 + k^2$

$\Rightarrow\ e^2h^2 + 2eah + a^2 = h^2 + 2eah + a^2e^2 + k^2$

$\Rightarrow\ h^2(1-e^2) + k^2 = a^2(1-e^2)$

$\Rightarrow\ \dfrac{h^2}{a^2} + \dfrac{k^2}{a^2(1-e^2)} = 1$

Hence the locus of (h, k) is

$\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1-e^2)} = 1$ **Ans. [A]**

Ex.11 The orthocentre of triangle with vertices $\left(2, \dfrac{\sqrt{3}-1}{2}\right), \left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$ and $\left(2, -\dfrac{1}{2}\right)$ is -

(A) $\left(\dfrac{3}{2}, -\dfrac{\sqrt{3}-3}{6}\right)$ (B) $\left(2, -\dfrac{1}{2}\right)$

(C) $\left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$ (D) $\left(\dfrac{5}{4}, \dfrac{\sqrt{3}-2}{4}\right)$

Sol. Here

AB $= \sqrt{\left(2-\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2} = \sqrt{3}$

BC $= \sqrt{\left(\dfrac{1}{2}-2\right)^2 + \left(-\dfrac{1}{2}+\dfrac{1}{2}\right)^2} = \dfrac{3}{2}$

CA $= \sqrt{(2-2)^2 + \left(-\dfrac{1}{2}-\dfrac{\sqrt{3}-1}{2}\right)^2} = \dfrac{\sqrt{3}}{2}$

Here $BC^2 + CA^2 = AB^2$

$\therefore$ ΔABC is right - angled triangle

Thus point C$\left(2, -\dfrac{1}{2}\right)$ is the ortho-centre

Ans. [B]

Ex.12 The number of points on x-axis which are at a distance c(c < 3) from the point (2, 3) is -

(A) 2 (B) 1

(C) infinite (D) no point

Sol. Let a point on x-axis is $(x_1, 0)$, then its distance from the point (2, 3)

$= \sqrt{(x_1-2)^2+9} = c$

or $(x_1 - 2)^2 = c^2 - 9$

$\therefore$ $x_1 - 2 = \sqrt{c^2-9}$

But $C < 3 \Rightarrow c^2 - 9 < 0$

$\therefore$ x_1 will be imaginary **Ans. [D]**